

Isotropic Lifshitz point in the $O(N)$ Theory

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The presence of an isotropic tricritical Lifshitz point for the $O(N)$ scalar theory is investigated in the $1/N$ expansion by means of the Functional Renormalization Group equations. At leading order, the non-trivial Lifshitz point is observed if the number of dimensions d is taken between $d = 4$ and $d = 8$, and the eigenvalue spectrum of the associated eigendirections is derived. At order $1/N$, the anomalous dimension η_N is computed and it is found to vanish both in $d = 4$ and $d = 8$, while it turns out to be positive between these two values, although strongly dependent on the choice of infrared regulator.

Introduction – The description of a tricritical Lifshitz point by a Landau-Ginzburg ϕ^4 model, where the derivatives of the field with respect to the space coordinates of a m -dimensional subset of a d -dimensional space and those of the complementary $(d - m)$ -dimensional subspace possess different scaling laws, was first presented in [1]. More specifically, in [1] the kinetic term with square gradient of the field, $O(\partial^2)$, is kept finite only for the second subset of coordinates, while the corresponding term of the m -dimensional subset is suppressed, so that the term with four powers of the gradient, $O(\partial^4)$, becomes the leading kinetic term of the m -dimensional subspace and this induces drastic changes in the scaling properties of the theory.

The Lifshitz points, which are related to the coexistence on the phase diagram of three phases, one with vanishing order parameter, another with finite constant order parameter and the third characterized by a modulated order parameter with finite wave vector, find application in various fields such as magnetic systems as well as polymer mixtures or high T_C superconductors (for reviews see [2, 3]), but, recently, also in different contexts such as Lorentz symmetry violation, [4, 5], or emergent gravity theories [6–9]. In addition, an oscillating phase has been predicted for a very wide class of systems [10–14], and it is conceivable to expect that a Lifshitz point could be associated to these modulated phases. In this sense, a more complete understanding of the properties of the Lifshitz point is certainly desirable.

Rather than considering the general case with $0 < m < d$, where the different scaling properties in the two separate subspaces lead to a peculiar critical behaviour that involves two different anomalous dimensions and two different correlation lengths, we shall focus on the isotropic case with $m = d$. In fact, if $m < d$, due to the different behaviour of the two sets of coordinates, the isotropy of the problem is lost while, when $m = d$, all the space coordinates have the same critical behaviour and spatial isotropy is preserved. Clearly, in this latter case the critical scaling remains different from the standard one because, as explained before, the kinetic term in the action is quartic, rather than quadratic, in the field derivatives.

The critical properties of the Lifshitz point were studied in the ε -expansion [1] as well as in the $O(1/N)$ expansion [15]. The isotropic case $m = d$ was considered within an expansion around $d = 8$ and $\varepsilon = 8 - d$ [16] while, recently, a numerical Monte-Carlo study indicated a possible disappearance of the

Lifshitz point, when fluctuations are properly taken into account [17].

Furthermore, another non-perturbative technique already employed to study this problem is the Functional Renormalization Group (FRG) [18–20] which consists of a set of differential flow equations either for various operators entering the effective action of the theory, or for one or more n -point Green functions derived from the effective action. Fixed points correspond to stationary points of these equations and the critical exponents, that classify relevant, marginal and irrelevant operators, are extracted by determining the eigenvalue spectrum of the linear reduction of the differential equations around the fixed point solutions. Coming to the Lifshitz point, the FRG was applied to study this problem for a one component scalar theory, $N = 1$, [21], and for the $N = 3$ theory, [22], both in the uniaxial ($m = 1$) case. Finally, the isotropic case ($m = d$) with $N = 1$ was considered in [23]. In particular in this last case, the Proper Time version [24–26] of the FRG, which can be formally derived in the framework of the background field flows [27, 28], was used because it proved to be quite accurate and suitable for the numerical analysis of the critical properties of a theory at a fixed point [26, 29–31] and, in addition, the Proper Time flow equation of the $O(\partial^4)$ operator (coupled to the potential and to the $O(\partial^2)$ operator equations), that is necessary to treat a Lifshitz point, had been already derived in [31].

The numerical analysis performed in [23] for the $N = 1$ theory, shows at the lowest order (in the Local Potential Approximation - LPA), i.e. by considering the fixed potential equation only, that a non-trivial solution exists when the number of spatial dimensions is $4 < d < 8$, and for $d \geq 8$ the solution merges with the trivial, gaussian fixed point, while for $d \leq 4$ the asymptotic structure of the differential equation changes and no discrete set of non-trivial solution is available. Then, when going beyond the LPA and including the differential equations for the $O(\partial^2)$ and $O(\partial^4)$ operators, a solution was observed in the range $5.5 < d < 8$, but the numerical analysis for smaller d becomes too demanding and it was not possible to establish whether the Lifshitz point survives down to $d = 4$ or, rather, the fluctuations associated with higher derivatives terms, $O(\partial^2)$ and $O(\partial^4)$, effectively destroy the critical behaviour when d approaches 4.

In this letter we consider another aspect of the problem and analyze the existence of a Lifshitz point for a scalar $O(N)$ -

symmetric theory, in order to find out whether the critical behaviour survives to the presence of the strong infrared fluctuations due to the transverse modes. To this aim, a numerical analysis would require the resolution of a very large number of differential equations that would probably present the same kind of problems observed for the simpler $N = 1$ case. Therefore, we prefer to rely on a different approach, introduced and developed in [32, 33], where the full flow equation for the effective action is projected onto a set of flow equations for the n -point Green functions which is then to be truncated at some specific n and, in our case, we shall consider the three equations for the potential and the longitudinal and transverse two-point functions. Then, these three equations are treated in the limit of large number of field components N , by explicitly expanding in powers of $1/N$. In this scheme we perform an analytical study which yields not only the complete eigenvalue spectrum of the Lifshitz point at the leading order, $1/N = 0$, but also the $1/N$ correction of the anomalous dimension at criticality.

Flow equations – In order to write down the fixed point equation, we start from the full FRG flow equation [20]: ($\partial_t \equiv k \partial_k$):

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_t R_k(q) \left[\Gamma_k^{(2)}[q, -q; \phi] + R_k(q) \right]^{-1} \quad (1)$$

$\Gamma_k[\phi]$ being the running effective action at scale k , and $R_k(q)$ a suitable regulator that suppress the modes with $q \ll k$ and allows to integrate those with $q \gg k$. The specific choice of the regulator $R_k(q)$ is discussed below.

Rather than introducing the running parameters by means of an explicit form of the effective action, we proceed by displaying the second functional derivative of the effective action $\Gamma_{ab}^{(2)}(p; \phi) \equiv \delta^2 \Gamma_k / (\delta \phi_a(p) \delta \phi_b(-p))$ that, according to the $O(N)$ symmetry of the theory, has the general form

$$\Gamma_{ab}^{(2)}(p, \phi) = \Gamma_A(p, \rho) \delta_{ab} + \phi_a \phi_b \Gamma_B(p, \rho) \quad (2)$$

with

$$\rho \equiv \frac{\phi_a \phi_a}{2} . \quad (3)$$

Then, we parametrize Γ_A and Γ_B in terms of the potential V and of the renormalization functions Z_A, Z_B, W_A, W_B , i.e. the coefficients of the quadratic and quartic powers of the momentum p :

$$\Gamma_A(p^2, \rho) = W_A(\rho) p^4 + Z_A(\rho) p^2 + V' \quad (4)$$

$$\Gamma_B(p^2, \rho) = N W_B(\rho) p^4 + N Z_B(\rho) p^2 + V'' \quad (5)$$

where prime indicates the derivative with respect to ρ and N is the number of field components.

The factor N appearing in front of W_B and Z_B is due to a specific rescaling of the potential and of the field with respect to the standard definitions, $V \rightarrow N V$ and $\phi_a \rightarrow \sqrt{N} \phi_a$, which

is made in order to derive a fixed point equation that is directly arranged in a $1/N$ expanded structure. Clearly, this rescaling has no effect on V' , while it changes $V'' \rightarrow V''/N$ as well as the factor $\phi_a \phi_b \rightarrow N \phi_a \phi_b$ in Eq. (2), these last two transformations being responsible for the factor N appearing in the definition of Γ_B in Eq. (5). Therefore, from Eqs. (4) and (5), it is easy to expect the parameters W_B, Z_B to be $1/N$ suppressed with respect to W_A, Z_A , as it will be checked below.

Then, if we separate the longitudinal (L) and transverse (T) components in the inverse of $\Gamma_{ab}^{(2)}(p, \phi)$, that is the propagator of the theory $G_{ab}(p, \phi)$, according to:

$$G_{ab}(p, \phi) = \left(\delta_{ab} - \frac{\phi_a \phi_b}{2\rho} \right) G_T(p, \rho) + \frac{\phi_a \phi_b}{2\rho} G_L(p, \rho) \quad (6)$$

one finds

$$G_T^{-1}(p, \rho) = \Gamma_A(p, \rho) \quad (7)$$

and

$$G_L^{-1}(p, \rho) = \Gamma_A(p, \rho) + 2\rho \Gamma_B(p, \rho) . \quad (8)$$

It is understood that the field dependent parameters V, W_A, W_B, Z_A, Z_B also depend on the running scale k and, with these settings, we can rely on the derivation of the flow equations carried out in [33]. We define the integrals

$$J_n^{\alpha\beta}(p, \rho) = \int_q \partial_t R_k(q) \tilde{G}_\alpha^{n-1}(q, \rho) \tilde{G}_\beta(p+q, \rho), \quad (9)$$

$$I_n^{\alpha\beta}(\rho) = J_n^{\alpha\beta}(0, \rho), \quad (10)$$

where $n \geq 1$, $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$, α and β stand either for L or T , and

$$(\tilde{G}_\alpha^n(q, \rho))^{-1} \equiv (G_\alpha^n(q, \rho))^{-1} + R_k(q) . \quad (11)$$

Then, by following [33] (see also [34]), we get the flow equation for the potential V :

$$\partial_t V(\rho) = \frac{1}{2} \left\{ I_1^{TT}(\rho) + \frac{1}{N} \left[I_1^{LL}(\rho) - I_1^{TT}(\rho) \right] \right\} \quad (12)$$

and for the two-point functions, properly subtracted of the zero-momentum contribution:

$$\partial_t [\Gamma_X(p^2, \rho) - \Gamma_X(0, \rho)] = F_X(p^2, \rho) - F_X(0, \rho) \quad (13)$$

where X stands either for A or B , and

$$F_A(p^2, \rho) = -\frac{1}{2} I_2^{TT} \Gamma'_A + \frac{1}{N} \left[2\rho \left(J_3^{LT} \Gamma_A'^2 + J_3^{TL} \Gamma_B'^2 \right) - I_2^{LL} \left(\frac{\Gamma'_A}{2} + \rho \Gamma_A'' \right) - I_2^{TT} \left(\Gamma_B - \frac{\Gamma'_A}{2} \right) \right], \quad (14)$$

$$F_B(p^2, \rho) = J_3^{TT} \Gamma_B'^2 - \frac{1}{2} I_2^{TT} \Gamma_B' + O\left(\frac{1}{N}\right) \quad (15)$$

Eqs. (13), (14), (15) can be reduced to flow equations either for W_X or Z_X , by selecting in F_X the terms proportional respectively to p^4 or p^2 . Then, it is evident from Eqs. (5), (13) and (15) that, in order to avoid any inconsistency in the $1/N$ expansion, W_B and Z_B must be $O(1/N)$ so that $\Gamma_B \sim O(1)$. Accordingly, we are allowed to neglect $O(1/N)$ corrections in Eq. (15), as they contribute to W_B and Z_B to order $O(1/N^2)$.

Let us now consider the regulator $R_k(q)$. A particularly useful regulator, that has the advantage of reducing the integrals to simple structures which can be analytically solved in most cases, was introduced in [35] and has the form:

$$R_k^\theta(q) = (k^2 - q^2) \widehat{Z}_k \theta(k^2 - q^2) \quad (16)$$

where θ is the Heaviside step function and a k -dependent (but field independent) normalization factor \widehat{Z}_k is included. For the present problem the regulator in Eq. (16) should be modified into $R_k^\theta(q) = (k^4 - q^4) \widehat{W}_k \theta(k^2 - q^2)$ with \widehat{W}_k taken equal to W_A , evaluated at a particular value of ρ : $\widehat{W}_k = W_A(\bar{\rho})$, with $\bar{\rho}$ to be specified. However, due to the presence of the Heaviside function, the second and higher derivatives of $R_k^\theta(q)$ with respect to q^4 , generate a singular behaviour of the integrals involved in this analysis. Therefore it is preferable to replace $R_k^\theta(q)$ with a smooth, one-parameter (α) regulator:

$$R_k(q) = \frac{\widehat{W}_k}{2} \left[(k^4 - q^4) + \sqrt{(k^4 - q^4)^2 + (2\alpha)^{-2}} \right] \quad (17)$$

In fact, $R_k(q)$ in Eq. (17) approaches $R_k^\theta(q)$ in the limit $1/\alpha \rightarrow 0$, and, for values of the dimensionless parameter $k^4 \alpha \sim 10^3$ or larger, $R_k(q)$ (and its first derivative) can be practically replaced by $R_k^\theta(q)$ in the resolution of the integrals but, on the other hand, all its derivatives are regular so that it does not generate any singularity as long as α is kept finite, i.e. $1/\alpha \neq 0$.

Incidentally, as the vanishing of the regulator at $k = 0$, $R_{k=0}(q) = 0$, is a necessary requirement of the flow equations, then $1/\alpha$ must be a function of the scale k that vanishes at $k = 0$ and this can be easily achieved by taking e.g. $1/\alpha = \lambda \Lambda^4 \exp(-\Lambda^4/k^4)$, Λ being a fixed scale and λ a small dimensionless parameter. However, in the analysis of the fixed point structure of the problem we look for k independent solutions of the flow equations and we are not interested in the limit $k \rightarrow 0$; therefore for our purposes α can be treated as a free parameter.

Leading order of the $1/N$ expansion – As anticipated, Eqs. (12), (13), (14) and (15), are already arranged in a $1/N$ expansion structure and we can straightforwardly extract the leading ($1/N = 0$) flow equations for the suitably rescaled parameters, and also the associated fixed point equations, which are obtained by requiring the rescaled parameters to be t -independent. The rescaled parameters, relevant for our analysis, are $\varrho = k^{-d+4-\eta} \rho$, $v = k^{-d} V$, $w^A = k^\eta W_A$, $w^B = k^{d-4+2\eta} W_B$, $z^A = k^{\eta-2} Z_A$, $z^B = k^{d-6+2\eta} Z_B$, where the scaling dimensions, i.e. the exponents in the powers of the scale k , are given in [22, 23], and the fixed point equations for w^A

and z^A at $1/N = 0$ are:

$$-\eta_0 w_0^A + (d - 4 + \eta_0) \varrho w_0^{A'} = -\frac{1}{2} I_2^{TT} w_0^{A'} \quad (18)$$

$$(2 - \eta_0) z_0^A + (d - 4 + \eta_0) \varrho z_0^{A'} = -\frac{1}{2} I_2^{TT} z_0^{A'} \quad (19)$$

In Eqs. (18) and (19) the prime indicates derivation with respect to ϱ and the subscript 0 indicates the lowest order of the $1/N$ expansion. It is easy to check that a field independent w_0^A (and therefore $w_0^{A'} = 0$) together with $\eta_0 = z_0^A = 0$ is a solution of this set of equations. Therefore, we can take $w_0^A = 1$ to set the overall normalization of the effective action.

Then, we turn to the fixed point equation for the potential, Eq. (12), and, after setting $\widehat{W}_k = 1$ in Eq. (17), the integral I_1^{TT} can be solved and Eq. (12) conveniently written as :

$$\left[(x + f(x))^2 d_+ - 1 \right] f(x) = \left[(x + f(x))^2 d_- - 1 \right] x f_x(x) \quad (20)$$

with the following definitions $x = \sqrt{2\varrho}$; $f(x) = dv/dx$; $f_x(x) = df/dx$; $d_\pm = (d \pm 4)/(2\tau)$ and finally $\tau = 2/[4\pi]^{d/2} \Gamma(1 + d/2)$ is the factor coming from the resolution of the integral I_1^{TT} .

Eq. (20) can be easily attacked numerically, but all the essential features can be deduced by simple inspection. In fact we immediately see that the constant function $f_G(x) = 0$ is a solution of Eq. (20), that plays the same role of the gaussian fixed point for the standard scaling. In addition, we observe that a viable non-trivial Lifshitz solution $f_L(x)$ must vanish at the origin $f_L(0) = 0$ due to the symmetry of the problem and, in addition, another zero of f_L must occur at

$$\bar{x}^2 = \frac{2\tau}{(d-4)} \quad (21)$$

i.e. $f_L(\bar{x}) = 0$ with non-vanishing derivative $f_{Lx}(\bar{x}) \neq 0$. By expanding Eq. (20) around \bar{x} , one finds from the linear terms:

$$f_{Lx}(\bar{x}) = \frac{8-d}{d-4} \quad (22)$$

Eq. (21) loses meaning when $d \leq 4$, while the vanishing of $f_{Lx}(\bar{x})$ from Eq. (22) at $d = 8$ indicates a flattening of the solution f_L onto the trivial solution f_G . The latter result accords with the numerical analysis of [23] with $N = 1$, which indicates that the two solutions merge at $d = 8$ and only the trivial solution survives for $d \geq 8$. Therefore we limit the study of Eq. (20) to the range $4 \leq d \leq 8$.

With the information collected above, we are able to determine the eigenvalues λ_L of the flow equation, linearized around the fixed point solution. To this aim we follow the procedure originally worked out in [36] for the standard Wilson-Fisher (WF) fixed point, (see also [30]) and, by writing the t -dependent function $f(t, x) = f_L(x) + e^{\lambda t} h(x)$, as the sum of the fixed point solution $f_L(x)$ and a perturbation $h(x)$, we get the the following linear (in $h(x)$) equation:

$$\begin{aligned} \frac{\lambda h}{\tau} = & \left[d_+ - (x + f_L)^{-2} \right] h - \left[d_- - (x + f_L)^{-2} \right] x h_x \\ & + 2(x + f_L)^{-3} (f_L h - x f_{Lx} h) \end{aligned} \quad (23)$$

The function h is supposed to be regular at any finite x and can be expanded around \bar{x} :

$$h(x) = \sum_{i=n}^{\infty} a_i (x - \bar{x})^i \quad (24)$$

where the lowest power n must be a non-negative integer, $n \geq 0$. At $x = \bar{x}$ the coefficient of $x h_x$ in square brackets vanishes and therefore $h_x(x)/h(x)$ is either singular at $x = \bar{x}$ (with a simple pole singularity) or finite, the former case corresponding to $n > 0$ and the latter to $n = 0$ in Eq. (24). In both cases, after dividing both members of Eq. (23) by h , one can make the replacement $h_x(x)/h(x) = n/(x - \bar{x})$ in order to compute the linear corrections in the expansion of Eq. (23) around the point \bar{x} . This expansion, with the help of Eq. (22), yields the following eigenvalue spectrum (we recall $4 \leq d \leq 8$) :

$$\lambda_L = d - 4 - 4n \quad (25)$$

parameterized by the non-negative integer $n \geq 0$. By following the same procedure, one derives from Eq. (23) the eigenvalues associated to f_G (again with integer $n \geq 0$) :

$$\lambda_G = 4 - (d - 4)n \quad (26)$$

In particular one can determine those values of n that correspond to relevant (positive) eigenvalues, namely $0 \leq n < (d - 4)/4$ from Eq. (25), and $0 \leq n < 4/(d - 4)$ from Eq. (26). In addition, we observe that in $d = 8$ the two spectra in Eq. (25) and (26) are equal, as the two fixed point solutions become coincident.

In conclusion, the solutions found at $1/N = 0$ with this particular scaling, clearly resemble those obtained with standard scaling where, aside from the constant gaussian solution with eigenvalue spectrum $\lambda_g = 2 - (d - 2)n$, one has the WF fixed point with $\lambda_{WF} = d - 2 - 2n$. One clearly sees that the difference, at this order, is only in the range spanned by d which, in this case, goes from $d = 2$ to $d = 4$, while, in the analysis of the tricritical Lifshitz point, from $d = 4$ to $d = 8$. In fact, even the number of relevant directions is the same in the two cases, once the proper change in d is taken into account.

$1/N$ corrections – At the leading order $1/N = 0$, the equations for the momentum dependent parts admit the elementary field-independent solutions $w_0^A = 1$ and $z_0^A = 0$, together with $\eta_0 = 0$, while the equations for w^B and z^B at this order decouple from the other equations and one is left with the fixed point equation for the potential only.

For the next step, we consider the potential expansion $v = v_0 + v_N/N + O(1/N^2)$ and the analogous expansions for η , w^A , z^A , w^B , z^B , and insert them into the fixed point equations in order to analyze the $1/N$ corrections. We start by observing that Eq. (12) for v_N involves the $(1/N)$ corrections of all the above variables (we recall here that $w_0^B = z_0^B = 0$, but the first non-vanishing terms of the expansion of w^B and z^B , which are $O(1/N)$, contribute to the leading order ($1/N = 0$) longitudinal propagator G_L , because of the factor

N in Eq. (5)). Therefore, a full determination of the $1/N$ corrections requires the resolution of five coupled equations.

However, it is possible to determine η_N without solving the whole set of equations. To this aim, a direct inspection of equations (13), (14), (15) shows that the vanishing field-independent solution $w_N^A = z_N^A = w_N^B = z_N^B = 0$ is not allowed because of the non-vanishing coefficients of the integrals J_3 in Eqs. (14), (15), respectively $\Gamma_A'^2$ and $\Gamma_B'^2$, which are finite and field dependent due to their dependence on V' and V'' , as shown in Eqs. (4), (5).

Nevertheless, at least for one particular value of the field $\varrho = \bar{\varrho}$, we can extend at $1/N$ the normalization of the propagators, already fixed by the leading order solution $w_0^A = 1$; $z_0^A = w_0^B = z_0^B = 0$. This immediately implies $w_N^A(\bar{\varrho}) = z_N^A(\bar{\varrho}) = w_N^B(\bar{\varrho}) = z_N^B(\bar{\varrho}) = 0$ and it is natural to take $\bar{\varrho}$ as the point where the derivative of the leading order potential vanishes, i.e. $\bar{\varrho} = \bar{x}^2/2$, with \bar{x} defined in Eq. (21). Finally, we extract from Eq. (14) the two equations for w_N^A , z_N^A , directly computed at $\bar{\varrho}$:

$$-\eta_N - (d - 4)\bar{\varrho} w_N^{A'}(\bar{\varrho}) = -\frac{1}{2}(I_2^{TT})_0 w_N^{A'}(\bar{\varrho}) + 2\bar{\varrho} v_0''(\bar{\varrho})^2 \left(J_3^{LT}|_{p^4} + J_3^{TL}|_{p^4} \right)_0 \quad (27)$$

$$-(d - 4)\bar{\varrho} z_N^{A'}(\bar{\varrho}) = -\frac{1}{2}(I_2^{TT})_0 z_N^{A'}(\bar{\varrho}) + 2\bar{\varrho} v_0''(\bar{\varrho})^2 \left(J_3^{LT}|_{p^2} + J_3^{TL}|_{p^2} \right)_0 \quad (28)$$

where the subscript 0 of the various integrals indicates that they must be computed by using the leading order ($1/N = 0$) solution of the various parameters, while the subscript p^4 in Eq. (27) and p^2 in Eq. (28) of the integrals J_3 , indicates that only the coefficient of that particular power of the momentum p in the expansion of the addressed integral is to be retained. We find that the $1/N$ correction to the anomalous dimension η_N does not appear in Eq. (28), but it is directly obtained from Eq. (27), if one neglects the terms proportional to $w_N^{A'}(\bar{\varrho})$.

At this point we observe that the procedure adopted to compute η_N essentially coincides with the scheme introduced in [37] which leads to the improved Local Potential Approximation LPA'. This can be straightforwardly checked by replacing Eq. (27) with the equation obtained by repeating the previous steps for the case of the anomalous dimension η_{WF} at the WF fixed point, which gives $\eta_{WF} = -2\bar{\varrho} v_0''(\bar{\varrho})^2 (J_3^{LT}|_{p^2} + J_3^{TL}|_{p^2})_0$. In this case the expansion is to be taken to order p^2 and the regulator in (16) can be safely chosen, because it does not generate any singularity. The corresponding integrals can be analytically computed, as shown in [37], and one finds $(J_3^{LT}|_{p^2} + J_3^{TL}|_{p^2})_0 = -\tau/(1 + 2\bar{\varrho} v_0''(\bar{\varrho}))^2$ and, therefore, $\eta_{WF} = 2\tau\bar{\varrho} v_0''(\bar{\varrho})^2/(1 + 2\bar{\varrho} v_0''(\bar{\varrho}))^2$. This is exactly the expression of the anomalous dimension which is used in the LPA' [20, 37, 38].

In order to test the reliability of this procedure, we can go one step further and replace in η_{WF} , the particular value

of $\bar{\varphi}$ and $v_0''(\bar{\varphi})$) that are obtained from the leading order analysis ($1/N = 0$) for the WF fixed point. Then, instead of Eq. (21), one has $\bar{x}^2 = \tau/(d-2)$ and Eq. (22) becomes $f_{WF,x}(\bar{x}) = 2\bar{\varphi}v_0''(\bar{\varphi}) = (4-d)/(d-2)$ (these changes are due to the different scaling of the various quantities in the two cases and also to the different dimension of the regulator R_k that, when derived with respect to the scale k , $\partial_t R_k$, produces a different factor). Thus, one finds the following $1/N$ correction to the anomalous dimension at the WF fixed point:

$$\eta_{WF} = \frac{(d-2)(4-d)^2}{4} \quad (29)$$

that is to be compared to the full result directly obtained in the $1/N$ expansion, [39] ($\epsilon \equiv 4-d$ and Γ indicates the Gamma function):

$$\eta = \frac{4\epsilon}{(4-\epsilon)\pi} \frac{\sin(\pi\epsilon/2) \Gamma(2-\epsilon)}{\Gamma(1-\epsilon/2) \Gamma(2-\epsilon/2)}. \quad (30)$$

Remarkably, Eqs. (29) and (30) have the same behaviour both for $d = 2 + \delta$ (with $\delta \gtrsim 0$), i.e. $\eta_{WF} = \eta = \delta$, and for $d \lesssim 4$ (with $\epsilon \gtrsim 0$), i.e. $\eta_{WF} = \eta = \epsilon^2/2$. Instead, in $d = 3$, where the difference between Eq. (29) and Eq. (30) is largest, one finds $\eta_{WF} = 1/4$ and $\eta = 8/(3\pi^2) \simeq 1/(3.7)$. This small discrepancy gives a measure of the reliability of the procedure here considered.

Going back to the Lifshitz fixed point problem, we have to compute η_N from Eq. (27) by neglecting the terms proportional to $w_N^{A'}(\bar{\varphi})$. However, as anticipated, this time a strong dependence on the regulator is observed. In particular, the parameter α introduced in Eq. (17) explicitly shows up in the resolution of the integrals, because $(\partial^2 R_k(q^4)/\partial q^8)_{q^4=k^4} = \alpha$. Namely, we get

$$\eta_N = \frac{4\tau \bar{\varphi} v_0''(\bar{\varphi})^2}{D^2} \left[4\bar{\alpha} - \frac{24\bar{\alpha} + d + 8}{(d+2)D} + \frac{6}{(d+2)D^2} \right] \quad (31)$$

where we introduced the dimensionless parameter $\bar{\alpha} = k^4 \alpha$ and $D = (1 + 2\bar{\varphi} v_0''(\bar{\varphi}))$. Then, with the help of Eqs. (21) and (22) one gets the analogous of Eq. (29) for the Lifshitz case:

$$\eta_N = \frac{(d-4)(8-d)^2}{16} \left\{ 4\bar{\alpha} - \frac{(d-4)}{4(d+2)} (24\bar{\alpha} + d + 8) + \frac{3(d-4)^2}{8(d+2)} \right\} \quad (32)$$

Therefore, the largest non-vanishing contribution to the anomalous dimension in the $1/N$ expansion, is derived with no need to solve the fixed point equation for the $O(1/N)$ correction to the potential, or the wave function renormalizations. We observe the explicit dependence on the parameter $\bar{\alpha}$ in Eq. (32) and it is evident that the alternative use of the Heaviside cutoff R_k^θ , associated to the limit $1/\bar{\alpha} \rightarrow 0$ would produce a singular behaviour of η_N . Instead, for finite values of $\bar{\alpha}$, one finds $\eta_N > 0$ in the whole range $4 < d < 8$, while at the

two critical values, $d = 4$ and $d = 8$, the anomalous dimension vanishes: $\eta_N = 0$.

Discussion – We analyzed the presence of the isotropic tricritical Lifshitz point for the $O(N)$ theory in the $1/N$ expansion and explicitly computed the anomalous dimension to order $1/N$. Already at leading order, the non-trivial Lifshitz point is observed only between $4 < d < 8$. At order $1/N$ the anomalous dimension η_N vanishes both at $d = 4$ and $d = 8$, being positive in between, and this is in agreement with the conjecture that these two values respectively represent the lower and upper critical dimension for the Lifshitz point of the $O(N)$ theory.

In particular, in [23], it is argued for the Lifshitz point of the $N = 1$ theory, that the lower critical dimension could be associated to the large field behaviour of the fixed potential, corresponding to the particular value of d below which the potential does no longer diverge as a power law for large values of the field ϕ but, instead, a continuous set of solutions (constant at large ϕ) of the fixed potential equation is found. This value of d is related to the change of sign of the scaling dimension of ϕ , that for the case considered is $D_\phi = (d - 4 + \eta)/2$ and therefore, if the anomalous dimension vanishes or is neglected, $d = 4$ is the requested value. For the Lifshitz point of the $O(N)$ theory where, as shown above, $\eta_N = 0$ in $d = 4$, it is natural to accept it as the lower critical dimension. Needless to say, this argument is the restatement of what occurs for the scaling of the $O(N)$ theory at the lower critical dimension of the WF fixed point, $d = 2$.

In addition, one can focus on the leading order potential equation, Eq. (20), directly in $d = 4$. Actually, this equation can be solved analytically and, as for the case with $d < 4$, one ends up with a continuous set of solutions, parameterized by one real parameter.

Finally, some comments on η_N displayed in Eq. (32) are in order. In fact, the strong regulator dependence, or $\bar{\alpha}$ -dependence, of this result has striking consequences. For instance in $d = 6$, one can take $\bar{\alpha}$ sufficiently large that the term η_N/N in the $1/N$ expansion of the anomalous dimension is so big, even with $N \gg 1$, that the expansion itself become questionable. Presumably, the choice of a much smoother cutoff could substantially reduce this problem, although it does not solve the issue of the strong regulator dependence in the determination of η_N . However, the above problem is drastically reduced in proximity of the two extremal values, $d = 4$ and $d = 8$, where the factor appearing outside the curly bracket in the right hand side of Eq. (32) assures the vanishing limit of η_N , when approaching either of these two points for any fixed value of $\bar{\alpha}$.

We conclude by observing that the tricritical Lifshitz point which, when looking at the eigenvalue spectrum at the leading order of the $1/N$ expansion, could appear as a trivial duplicate of the WF fixed point with a suitable redefinition of the scaling dimensions of the various operators, does actually show original features. In fact, not only rather different properties of the anomalous dimension (with respect to the WF case) show up

at order $1/N$, but it must also be noticed that, as soon as the wave function renormalizations are explicitly included in the fixed point equations, the coefficient of $(\partial\phi)^2$, Z , has positive scaling dimension $2-\eta$, which indicates the existence of a relevant direction that has no correspondence at the WF critical point. On the other hand the similarities in the two cases could be a hint that the structure observed around $d = 2$, such as the presence of multi-critical solutions [38, 40–43], or the relation with phase transitions of different nature [44, 45], could have a counterpart in the Lifshitz scaling around $d = 4$.

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